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Structural Characterization of Compoundness[†]

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We recover the rays in the tensor product of Hilbert spaces within a larger class of so-called 'states of compoundness', structured as a complete lattice with the 'state of separation' as its top element. At the base of the construction lies the assumption that the cause of actuality of a property of one individual entity is the actuality of a property of the other.

1. INTRODUCTION

Most approaches toward a realistic description of compound—quantum—systems are based on the recognition of subsystems, imposing some mathematical universal property as a structural criterion [6, 8, 9, 13, 14, 25, 27]. In this paper we take a different point of view, essentially focusing on a structural characterization of the interaction between the individual entities, rather than on the compound system as a whole. More precisely, we structurize the concept of 'mutual induction of actuality' for the 'individual entities' in a compound system, inspired by the existence of an—essentially unique—representation for compound quantum systems when postulating that a state transition of one individual entity induces a state transition of the others [20]. In particular we will consider 'separation' as one particular state of compoundness, and not as a type of 'entity' as in Aerts [11] and Ischi [25], as such avoiding some axiomatic drawbacks that emerge when taking the latter perspective. Formally, an essential ingredient of the reasoning can be recuperated from Faure *et al.* [18], where propagation of states and properties in maximal deterministic evolutions is studied. As an application of our way of looking at compoundness we mention a representation for spin

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systems [17, 21]. The mathematical preliminaries to this paper are basic notions on linear operators, for which we refer to Weidmann [10], and that of a Galois dual pair, i.e., a couple of isotone maps $f: M \to N$ and $f^*: N \to M$ satisfying $\forall x \in M, y \in N: x \leq f^*(y) \Leftrightarrow f(x) \leq y$, where f preserves existing joins, f^* preserves existing meets, and we have existence and uniqueness of it for a meet (resp. join)-preserving map between complete lattices [1, 12].

2. COMPOUNDNESS AND ASSIGNMENT OF CAUSES

Let us first recall the general concept of an *entity*, along the lines of refs. 5, 11, and 28; we will not go into the details on this and refer for the most recent overview to Moore [24]. We consider an entity to be a physical system described by a collection of properties $\mathcal{L}$ partially ordered by an—operationally motivatable—implication relation $\leq$, and *proves* to be a complete lattice [5]. As is discussed in Piron [7] and Moore [24], the meet $\wedge$ can be treated as a classical conjunction. A property is said to be 'actual' if we get *true* with certainty when it would be verified in any possible way; $a \in \mathcal{L}$ is stronger (resp. weaker) than $b \in \mathcal{L}$ iff $a \leq b$ (resp. $b \leq a$); the top element 1 of the complete lattice can be seen as expressing 'existence' of the entity and the bottom element 0 expresses the 'absurd', i.e., what can never be true. As an example, the property lattice $\mathcal{L}_{\mathcal{H}}$ of a quantum entity described in a Hilbert space $\mathcal{H}$ is the set of closed subspaces ordered by inclusion with intersection as meet and closed linear span as join. We will also systematically use the term 'individual entity' when considering identifiable 'parts' in a larger system—as such to be seen as a compound system—since in general, these individual entities do not satisfy the general conception of what an entity is in the references mentioned above—for a discussion on this aspect see Coecke [20]. In the presence of interaction between individual entities, the actuality of a property $a_2 \in \mathcal{L}_2$ of an individual entity S_2 might be due to the actuality of a property $a_1 \in \mathcal{L}_1$ of individual entity S_1. In particular, we will show that all interaction involved in quantum entanglement can be expressed in this way, and therefore we define a map:

$$f: \quad \mathcal{L}_2 \to \mathcal{L}_1: \quad a_2 \mapsto \text{“the cause in } \mathcal{L}_1 \text{ of the actuality of } a_2\text{”} \qquad (1)$$

This cause of actuality of a_2 is the *weakest* $a_1 \in \mathcal{L}_1$ *that assures actuality of* a_2: indeed, any $b_1 \in \mathcal{L}_1$ with $b_1 \leq a_1$ then automatically causes actuality of a_2 since it implies a_1. Note that existence of such a weakest $a_1 \in \mathcal{L}_1$ follows from the fact that 'assuring actuality of a_2' precisely defines it as a property for S_1. As an example, when considering two separated individual entities it is the bottom element $0_1 \in \mathcal{L}_1$ that assures actuality of any $a_2 \in \mathcal{L}_2 \backslash \{1_2\}$, explicitly expressing that no state of $\mathcal{L}_1$ assures anything about $\mathcal{L}_2$. On the contrary, the property 1_1 assures actuality of 1_2 since we *a priori* assume the existence of both individual entities. When considering meets $\wedge_i a_{2,i}$ in $\mathcal{L}_2$, due to their significance as a classical conjunction, i e., they can be read as 'and', $\wedge_i f^*(a_{2,i})$ assures actuality of $\wedge_i a_{2,i}$, as such assuring f^* to preserve nonempty meets. Since $\wedge\emptyset = 1$ and $f^*(1_2) = 1_1$ by assumption of the existence of both individual entities, it also preserves the empty meet. Thus, there exists a unique join-preserving Galois dual for f^*, namely

$$f\colon \quad \mathcal{L}_1 \to \mathcal{L}_2\colon \quad a_1 \mapsto \wedge\{a_2 \in \mathcal{L}_2 | a_1 \leq f^*(a_2)\}$$
$$= min\{a_2 \in \mathcal{L}_2 | a_1 \leq f^*(a_2)\} \qquad (2)$$

From Eq. (2) it follows that f assigns to a property $a_1 \in \mathcal{L}_1$ the strongest property $a_2 \in \mathcal{L}_2$ of which it assures actuality—the minimum of all $a_2 \in \mathcal{L}_\in$ such that $a_1 \leq f^*(a_2)$—implicitly implying actuality of all $b_2 \geq a_2$. Thus, f expresses exactly *induction of actuality* of $\mathcal{L}_1$ *on* $\mathcal{L}_2$.

Conclusion 1. A 'state of compoundness for S_1 on S_2' is a join-preserving map $f\colon \mathcal{L}_1 \to \mathcal{L}_2$.

2.1. Structuring States of Compoundness

Denote by $\mathfrak{Q}(\mathcal{L}_1, \mathcal{L}_2)$ the join-preserving maps from $\mathcal{L}_1$ to $\mathcal{L}_2$ and set for all $\{f_i\}_i \subseteq \mathfrak{Q}(\mathcal{L}_1, \mathcal{L}_2)$

$$\bigvee_i f_i := \mathcal{L}_1 \to \mathcal{L}_2\colon \qquad a \mapsto V_i\, f_i(a) \qquad (3)$$

Equation (3) defines a complete internal operation on $Q(\mathcal{L}_1, \mathcal{L}_2)$ since for $\{f_i\}_i \subseteq Q(\mathcal{L}_1, \mathcal{L}_2)$: $(\wedge_i f_i)(\wedge_j a_j) = \vee_{ij} f_i(a_j) = \vee_j (\vee_i f_i)(a_j)$, and one easily verifies that $\vee_i f_i$ is the *lub* of $\{f_i\}_i$ in $Q(\mathcal{L}_1, \mathcal{L}_2)$.

Conclusion 2. The states of compoundness for S_1 to S_2 are described by a complete lattice $(\mathfrak{Q}(\mathcal{L}_1, \mathcal{L}_2), \vee)$, inheriting its join from the underlying property lattice $\mathcal{L}_2$ pointwisely.

Analogously, the collection of meet-preserving maps $f^*\colon \mathcal{L}_2 \to \mathcal{L}_1$ denoted by $Q^*(L_1, L_2)$ is a meet complete lattice with respect to pointwise meet denoted as $\wedge$. Note here that the significance of the lattice meet as a classical conjunction is lifted by the pointwise computed meets to the level of assignment of temporal causes, as such giving to $f^* \wedge g^*$ the significance of 'f^* and g^*'. Set $L^*(f, a_2) = \{a_1 \in \mathcal{L}_1 | f(a_1) \leq a_2\}$ and $L(f^*, a_1) = \{a_2 \in \mathcal{L}_2 | a_1 \leq f^*(a_2)\}$. If $\forall a_1 \in \mathcal{L}_1$: $f(a_1) \leq g(a_1)$, then $[g(a_1) \leq a_2 \Rightarrow f(a_1) \leq a_2]$ and $\forall a_2 \in \mathcal{L}_2$: $L^*(g, a_2) \subseteq L^*(f, a_2)$. Thus, $\vee L^*(g, a_2) \leq \vee L^*(f, a_2)$, yielding $\forall a_2 \in \mathcal{L}_2$: $g^*(a_2) \leq f^*(a_2)$ and $g^* \leq f^*$. Conversely, $\forall a_2 \in \mathcal{L}_2$: $g^*(a_2) \leq f^*(a_2)$ implies $[g^*(a_2) \geq a_1 \Rightarrow f^*(a_2) \geq a_1]$, $L(g^*, a_1) \subseteq L(f^*, a_1)$, $\wedge L(g^*, a_1)$, $\geq \wedge L(f^*, a_1)$, and thus $f \leq g$. As such $f \leq g$ iff $g^* \leq f^*$.

It follows that $(Q^*(\mathscr{L}_1, \mathscr{L}_2), \wedge)^{op}$ and $(Q(\mathscr{L}_1, \mathscr{L}_2), \vee)$ are isomorphic complete lattices, where *op* stands for reversal of partial order.

Conclusion 3. The map $*: Q(\mathscr{L}_1, \mathscr{L}_2) \to Q^*(\mathscr{L}_1, \mathscr{L}_2): f \mapsto f^*$ 'interprets' $\vee_i f_i$, the join of a set of states of compoundness, as $\wedge_i f_i^*$ a conjunction of the corresponding assignments of temporal causes.

2.2. Separation as the Top State of Compoundness

Since we have $f^*: \{\mathscr{L}_2 \backslash \{1_1\} \to \mathscr{L}_1: a_2 \mapsto 0_1;\ 1_2 \mapsto 1_1\}$ in case of separation, the 'state of separation' for S_1 on S_2 is given by $f: \{\mathscr{L}_1 \backslash \{0_2\} \to \mathscr{L}_2: a_1 \mapsto 1_2; 0_1 \mapsto 0_2\}$ and this is exactly the top element $1_{1,2}$ of $\mathfrak{Q}(\mathscr{L}_1, \mathscr{L}_2)$; note that $0_1 \mapsto 0_2$ is required for any join-preserving map, the bottom being the empty join. Remark that the bottom element $0_{1,2}$ of $\mathfrak{Q}(\mathscr{L}_1, \mathscr{L}_2)$ stands for the 'absurd state of compoundness'. Indeed, since $f: \mathscr{L}_1 \to \mathscr{L}_2: a_1 \mapsto 0_1$ we have $f^*: \mathscr{L}_2 \to \mathscr{L}_1: a_2 \mapsto 1_1$, i.e., existence of S_1 causes actuality of all properties of S_2, and as such actuality of the absurd property $\wedge\mathscr{L}_2 = 0_2$.

2.3. Quantum Entanglement as Atomic States of Compoundness

We will now consider those $f \in \mathfrak{Q}(\mathscr{L}_{\mathscr{H}_1}, \mathscr{L}_{\mathscr{H}_2})$ that send atoms to atoms or 0_2 for $\mathscr{L}_{\mathscr{H}_1}$ and $\mathscr{L}_{\mathscr{H}_2}$ the lattices of closed subspaces of Hilbert spaces. The following result can be found in Faure *et al.* [18] and is essentially based on Faure and Frölicher [15, 16] and Piron [3, 5]: *Any nontrivial*[3] $f \in \mathfrak{Q}(\mathscr{L}_{\mathscr{H}_1}, \mathscr{L}_{\mathscr{H}_2})$ *that sends atoms to atoms or* 0_2 *induces either a linear or an antilinear map* $F: \mathscr{H}_1 \to \mathscr{H}_2$.

Proposition 1. If $f \in \mathfrak{Q}(\mathscr{L}_{\mathscr{H}_1}, \mathscr{L}_{\mathscr{H}_2})$ sends atoms to atoms or 0_2, then f is itself an atom or $0_{1,2}$.

Proof. Let K_f and K_g be the respective kernels of the linear maps F, G: $\mathscr{H}_1 \to \mathscr{H}_2$ induced by $f, g \in \mathfrak{Q}(\mathscr{L}_{\mathscr{H}_1}, \mathscr{L}_{\mathscr{H}_2})$ with $f < g$, and thus, $K_g \subset K_f$ and $K_f^\perp \subset K_g^\perp$. For $\psi \in K_f^\perp \backslash \{\underline{0}_1\}$ and $\phi \in (K_f \cap K_g^\perp) \backslash \{\underline{0}_1\}$ we have $F(\psi + \phi) = F(\psi) = G(\psi)$, whereas $G(\psi + \phi) = G(\psi) + G(\phi)$, forcing $G(\psi) = kG(\phi)$ for k nonzero. However, then $G(k\psi - \phi) = 0$ although $k\psi - \phi \in K_g^\perp$, yielding contradiction except when $K_f^\perp = \underline{0}_1$, i.e.,$[f: a_1 \mapsto 0_2] = 0_{1,2}$. ■

For the sake of transparency of the argument we will from now on only consider finite-dimensional Hilbert spaces. Let $\mathscr{H}_1'$ be the Hilbert space of continuous linear functionals on $\mathscr{H}_1$, connected to it by the correspondence $\mathscr{H}_1 \to \mathscr{H}_1': \psi \mapsto \langle\psi|-\rangle$. Then the tensor product $\mathscr{H}_1' \otimes \mathscr{H}_2$ is isomorphic to

[3] I.e., with at least three noncollinear elements in its range. Note that if the image is spanned by either one or two atoms there is an obvious representation as a linear map, extending the collection of representations in a natural way, which motivates us to assume a linear representation on the underlying Hilbert space for any state of compoundness.

the space of linear operators with indicated domain and codomain—denoted as $B(\mathcal{H}_1, \mathcal{H}_2)$—by the isomorphism:

$$\mathcal{H}_1' \otimes \mathcal{H}_2 \to B(\mathcal{H}_1, \mathcal{H}_2): \quad \left[\sum_{i=1}^{m} c_i\langle\psi_i|-\rangle \otimes \phi_i\right]$$
$$\mapsto \left[F_{\{c_i\}_i}: \mathcal{H}_1 \to \mathcal{H}_2: \psi \mapsto \sum_{i=1}^{m} c_i\langle\psi_i|\psi\rangle\phi_i\right] \quad (4)$$

with $\{\psi_i\}_i$ and $\{\phi_i\}_i$ fixed orthonormal bases; note that

$$\|F_{\{c_i\}_i}\|_{\mathrm{HS}} = \left\|\sum_{i=1}^{m} c_i\langle\psi_i|-\rangle \otimes \phi_i\right\|_{\mathcal{H}_i'\otimes\mathcal{H}_2}$$

for $\| - \|_{\mathcal{H}_1'\otimes\mathcal{H}_2}$ the Hilbert space metric and $\| - \|_{\mathrm{HS}}$ the Hilbert–Schmidt norm. Considering antilinear maps $\Sigma_{i=1}^{m} c_i\langle -|\psi_i\rangle\phi_i$, say $B'(\mathcal{H}_1, \mathcal{H}_2)$, and reasoning along the same lines, we obtain $\mathcal{H}_1 \otimes \mathcal{H}_2$. As such, we recover the rays of the tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ as a special case of our more general class of atomic states of compoundness, besides the rays $\mathcal{H}_1' \otimes \mathcal{H}_2$. Indeed, although $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}_1' \otimes \mathcal{H}_2$ are isomorphic as Hilbert spaces, there rays represent different states of compoundness.

3. COMPOUNDNESS AS EVOLUTION AND VICE VERSA

Although we were able to use much material from Faure *et al.* [18] dealing with the evolution of an entity, we should note that an essential difference between the two formal developments is due to the fact that existence of the entity at a certain instance of time in general does not assure existence in the future when considering evolution. However, imposing a condition on the 'type' of evolution that we consider by 'requiring preservation' makes an illustrative comparison possible, and as such we will proceed. To fix ideas, identify $\mathcal{L}_1$ with the property lattice of a fixed entity at time t_1 and $\mathcal{L}_2$ as its property lattice at time t_2. We can again consider a map f^*: $\mathcal{L}_2 \to \mathcal{L}_1$ that assigns to a property that is actual at time t_2 the cause of its actuality at time t_1, and the corresponding Galois dual f: $\mathcal{L}_1 \to \mathcal{L}_2$ that now expresses temporal propagation of properties, all of them again structured in $\mathfrak{Q}(\mathcal{L}_1, \mathcal{L}_2)$.

Example 1. A 'maximal deterministic evolution' is defined as one where atoms propagate to atoms or 0_2—the latter in order to express a domain for initial states—mirroring atomic states of compoundness.

Example 2. We could define a 'maximal indeterministic—or minimal deterministic—evolution' as one that strictly assures existence at time t_2, yielding the mirror of the state of separation.

For a more elaborate formal discussion on the connection between compoundness and evolution we refer to ref. 22 and in particular ref. 26.

4. ON ASSIGNMENT OF PROPER STATES TO INDIVIDUALIZED ENTITIES

The discussion on evolution was not merely illustrative, but constitutes an essential ingredient in this section, where we discuss state transitions of individual entities due to a state of compoundness: in Section 2 we characterized entanglement between individual entities in a compound system, but no consideration of a characterization of individual entities themselves has been made. Indeed, a description of a compound system consisting of two individual entities S_1 and S_2 requires a characterization of these individual entities themselves besides the states of compoundness $f_{1,2}$ of S_1 on S_2 and $f_{2,1}$ of S_2 on S_1. However, this requires a more general concept of state than in refs. 5, 11, and 19, and a different name, *proper state*, has been introduced [20] to stress this difference. Therefore consider for each individual entity an *a priori* set of proper states Σ and denote by $\mathscr{C}: \mathscr{P}(\Sigma) \to \mathscr{L}$ the map that assigns to any T in $\mathscr{P}(\Sigma)$, the powerset of Σ, the strongest property in $\mathscr{L}$ that is implied by every $p \in T$. As shown in refs. 22 and 23, $\mathscr{C}$ canonically induces a preorder on Σ by $p \leq_{c}, q \Leftrightarrow \mathscr{C}(\{p\}) \leq \mathscr{C}(\{q\})$. Moreover, the above requirement that properties propagate with the preservation of join (see also ref. 4) restricts the—not-necessarily deterministic—state transitions to be described by a map in

$$Q^{\#}(\Sigma) = \{f: \mathscr{P}(\Sigma) \to \mathscr{P}(\Sigma) \,|\, f(\cup T) = \cup f(T), f(\mathscr{C}(T)) \subseteq \mathscr{C}(f(T))\} \quad (5)$$

which proves to be a quantale $(Q^{\#}, (\Sigma), \cup, \circ)$, i.e., a join-complete lattice equipped with an additional operation $\circ$ that distributes over arbitrary joins, where $\cup$ is computed pointwise, and which is in epimorphic—quantale—correspondence with the 'quantale' $(Q\,(\mathscr{L}, \mathscr{L}), \vee, \circ)$ (see above) by

$$\bigvee[-]: \quad Q^{\#}\,(\Sigma) \to Q(\mathscr{L}, \mathscr{L}): \quad f \mapsto [\,f_{\mathscr{L}}: \mathscr{C}(T) \mapsto \mathscr{C}(f(T))] \qquad (6)$$

Note that this quantale epimorphism indeed exactly expresses that to any state transition there corresponds a join-preserving propagation of properties. We apply all this to the context of this paper:

- For each individual entity S, let $\Psi: \mathscr{L} \to Q^{\#}\,(\Sigma)$ be the map that assigns to any property $a \in \mathscr{L}$ the—not-necessarily deterministic—state transition $\Psi(a): \mathscr{P}(\Sigma) \to \mathscr{P}(\Sigma)$ that S undergoes when actuality of property a is induced on it by interaction with another individual entity.

Clearly this implicitly determines the propagation of properties $\Psi_{\mathscr{L}}: \mathscr{L} \to Q(\mathscr{L}, \mathscr{L})$ by $\vee[\Psi(-)] = \Psi_{\mathscr{L}}(-)$. We will now discuss Ψ and $\Psi_{\mathscr{L}}$ for $\mathscr{L}$ orthomodular, and more specifically, for the Hilbert space case.

4.1. Ordering of States via 'Descending' Induction of Actual Properties

Proposition 2. If the following conditions are satisfied, then $p \trianglelefteq q \Leftrightarrow [\exists a \leq \mathscr{C}(\{q\}): p \in \Psi(a)(\{q\})]$ defines a poset $(\Sigma, \trianglelefteq)$ that embeds in $(\Sigma, \leq_{\mathscr{C}})$:

(i) $\Psi(a)$ does not alter proper states of which the strongest actual property is stronger than a.

(ii) All properties compatible with the induced one that are actual beforehand remain actual.

(iii) The assignment $\mathrm{im}(\Psi_{\mathscr{L}}) \to \mathrm{im}(\Psi)$: $\Psi_{\mathscr{L}}(a) \mapsto \Psi(a)$ preserves composition, with $\mathrm{im}(-) = $ image.

Proof. Following Piron [5, p. 69, Theorem 4.3], if properties described by an orthomodular property lattice change in such a way that a property $a \in \mathscr{L}$ becomes actual, and such that all properties compatible with a that were actual beforehand are still actual afterward, then the corresponding transition of properties is exactly described by the Sasaki projection φ_q: $\mathscr{L} \to \mathscr{L}$: $b \mapsto a \wedge (b \vee a^{\perp})$. Thus, (ii) assures that $\Psi_{\mathrm{L}}(a) = \varphi_a$. If $p \in \Psi(a)(\{q\})$ for $a \leq \mathscr{C}(\{q\})$, then $\mathscr{C}(\{p\}) \leq \mathscr{C}(\Psi(a)(\{q\})) = \varphi_a\,(\mathscr{C}(\{q\}) \leq a$ [the equality follows from Eq. (6)], we have $\mathscr{C}(\{p\}) \leq \mathscr{C}(\{q\})$ and thus $p \leq_c q$. Note that (i) is equivalent to $\Psi(a)$ being identical on all $p \in \Sigma$ such that $\mathscr{C}(\{p\}) \leq a$, and thus yields reflexivity since it forces the restriction of $\Psi(a)$ to $\{T \subseteq \Sigma | \mathscr{C}(T) \leq a\}$, which is the only part of the domain involved in defining $\trianglelefteq$, to be idempotent. We cannot have $p \triangleleft q$ and $q \triangleleft p$ since any transition $\Psi(a)(\{q\}) = \{p\}$ requires again by (i) that $\mathscr{C}(\{p\}) \not\leq a$ where $\mathscr{C}(\{q\}) \leq a$. Following Foulis [2], the maps $\{\varphi_a | a \in \mathscr{L}\}$ are structured in a complete lattice isomorphic to $\mathscr{L}$ itself when ordered by $\varphi_a \leq \varphi_a \Leftrightarrow \varphi_{a'} = \varphi_{a'}\varphi_a$. Now consider $a \geq a'$; then $\varphi_a \geq \varphi_{a'} = \varphi_{a'}\varphi_a$, i.e., $\varphi_{a'}\varphi_a$ only depends on a' and not on a. Thus, (iii) yields transitivity since it forces $\Psi(a')(\Psi(a)(\{q\})) = \Psi(a')(\{q\})$, and this independent of a provided that $a \geq a'$. ∎

Note that condition (ii) says that actual properties are only altered in a minimal way. Now consider, under the assumptions of the above proposition, a chain $a_1 \geq a_1' \geq \ldots$ in $\mathscr{L}_1$ of consecutive strongest actual properties of S_1. By isotonicity of $f_{1,2}$ we have $f_{1,2}(a_1) \geq f_{1,2}(a_1') \geq \ldots$, and thus for $\{p_2'\} = \Psi(f_{1,2}(a_1'))(\{p_2\})$ we have $f_{1,2}(a_1) = \mathscr{C}(\{p_2\}) \geq \mathscr{C}(\{p_2'\}) \geq \ldots$ for p_2 initial, yielding $p_2 \trianglerighteq p_2' \trianglerighteq \ldots$. As such, the relation $\trianglelefteq$ expresses evolution of S_2 due to mutual induction of actuality as a *descending chain of proper states in* Σ_2*, with descending strongest actual properties in* $\mathscr{L}_2$*.*

4.2. Proper States for Compound Quantum Systems

At this point it is required to propose a candidate for the sets of proper states Σ_1 and Σ_2 in the Hilbert space case that allows us to fully recover the description of compound quantum systems. Let $P_{a_i}: \mathcal{H}_i \to \mathcal{H}_i$ be the orthogonal projector corresponding to φ_{a_i} for $i \in \{1, 2\}$ with a_i a closed subspace of $\mathcal{H}_i$ and set

$$\begin{cases} (\Sigma_i, \mathcal{C}_i) = (\{\rho_i: \mathcal{H}_i \to \mathcal{H}_i | \rho_i \text{ is a density operator}\}, \quad \mathcal{P}(\Sigma_i) \to \mathcal{L}_{\mathcal{H}_i}: \{\rho_i\} \mapsto \{\rho_i(\phi) | \phi \in \mathcal{H}_i\}) \\ \qquad \Psi_i(a_i) = \left[\mathcal{C}_i(\rho_i) \perp a_i: \{\rho_i\} \mapsto \emptyset; \mathcal{C}_i(\rho_i) \not\perp a_i: \{\rho_i\} \mapsto \left\{ \dfrac{1}{Tr(P_{a_i}\rho_i P_{a_i})} P_{a_i}\rho_i P_{a_i} \right\} \right] \end{cases} \tag{7}$$

One easily verifies that Ψ_i defines state transitions for $(\Sigma_i, \mathcal{C}_i)$ that fulfill Proposition 2 and that the one-dimensional projectors are minimal in $(\Sigma_i, \trianglelefteq)$. Moreover, the restriction of $\Psi_i(a_i)$ to $\{T \subseteq \Sigma_i | \mathcal{C}_i(T) \leq a_i\}$ maps a singleton on a singleton or $\emptyset$ for all $a_i \in \mathcal{L}_i$, i.e., the transitions due to induction of properties less than the strongest actual one is maximally deterministic. Now, given $F \in B(\mathcal{H}_1, \mathcal{H}_2)$ representing $f_{1,2}$ with $F^\dagger$ as its adjoint, then

$$\begin{aligned} F \mapsto \Bigg(& F: \mathcal{H}_1 \to \mathcal{H}_2, \frac{1}{Tr(F^\dagger F)} F^\dagger F : \mathcal{H}_1 \to \mathcal{H}_1, \\ & \frac{1}{Tr(FF^\dagger)} FF^\dagger: \mathcal{H}_1 \to \mathcal{H}_1, F^\dagger: \mathcal{H}_2 \to \mathcal{H}_1 \Bigg) \end{aligned} \tag{8}$$

uniquely defines a quadruple $(f_{1,2}, \rho_1, \rho_2, f_{2,1})$ which exactly yields the quantum probability structure in the following way: (i) The transition probability for $\{\rho_i\} \mapsto \{\rho_i'\} = \Psi_i(a_i)(\{\rho_i\})$ with $a_i \not\perp \mathcal{C}_i(\{\rho_i\})$ in a measurement that satisfies the property a_i is $Tr(P_{a_i}\rho_i P_{a_i})$; (ii) when this happens, say $\{\rho_1\} \mapsto \{\rho_1'\}$, this transition causes $a_1' = \mathcal{C}_1(\{\rho_1'\})$ to become actual, and consequently causes $a_2' = f_{1,2}(a_1')$ to become actual, having a transition $\{\rho_2\} \mapsto \{\rho_2'\} = \Psi_2(a_2')(\{\rho_2\})$ as a consequence; (iii) this reasoning can proceed inductively, and stops once we reach the minimal elements of $(\Sigma_2, \trianglelefteq)$. It can then be verified that the probability for the chains $\rho_1 \geq \rho_1' \geq \ldots \geq P_\psi$ and $\rho_2 \geq \rho_2' \geq \ldots \geq P_\psi$ to have a couple $(\mathcal{C}_1(P_\phi), \mathcal{C}_2(P_\phi))$ as respective outcome 'states', which are indeed represented as atoms of the property lattice, is given by[4]

[4] A proof follows from identification of this construction with the the representation for compound quantum systems in ref. 20: our choice of $(\Sigma_i, \mathcal{C}_i)$ and the corresponding quadruples $(f_{1,2}, \rho_1, \rho_2, f_{2,1})$ for linear maps F coincide exactly.

$$\frac{1}{\|\psi\|^2_{\mathcal{H}_1}\,\|\phi\|^2_{\mathcal{H}_2}\|F\|^2_{HS}}\left|\langle\psi\otimes\phi\right|\sum_i c_i\psi_i\otimes\phi_i\rangle\Big|^2 \tag{9}$$

for $F = \sum_{i=1}^m c_i\langle -|\psi_i\rangle\phi_i$, and this is indeed the quantum transition probability in a measurement on a compound system described by $\Sigma_i\ c_i\psi_i \otimes \phi_i \in \mathcal{H}_i \otimes \mathcal{H}_2$. We end by stressing that due to the assignment in (8), the initial proper states of the compound quantum system are fully encoded in the states of compoundness, and thus encoded in $\mathcal{H}_1, \otimes \mathcal{H}_2$ via B' $(\mathcal{H}_1, \mathcal{H}_2)$.

5. CONCLUSION

In this paper we proposed an alternative approach toward an understanding of the description of compound quantum systems by essentially focusing on the interaction of the individual entities within the compound system. Obviously, much more investigation could be done on a more accurate characterization of quantum entanglement as a special case of the primal considerations made in this paper. Also, an elaboration on the description of compound systems consisting of more than two entities is worthwhile.

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